Laminar Boundary Layer behind a Shock with Vaporization and Combustion

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The laminar boundary layer behind a normal shock wave with vaporization at the wall and combustion within the boundary layer is formulated for Prandtl and Schmidt numbers equal to 1. The problem is reduced to the Blasius equation with new boundary conditions. The solutions, which extend the work of Blasius, Emmons, and Mirels, cover a wide range of shock Mach numbers and wall material-gas combinations. Boundary-layer profiles which were obtained on an analog computer are presented.

Nomenclature

A = constant defined by Eq. (7); scale factor (Fig. 2)
B = constant defined by Eq. (28)
C_D = mean drag coefficient
C_H = mean heat-transfer coefficient
C_M = mean mass transfer coefficient
C_p = specific heat at constant pressure
D = binary diffusion coefficient
f = Blasius function defined by Eq. (22)
h = specific enthalpy = $\Sigma Y_i h_i$
h_i° = standard heat of formation per unit mass of species i at temperature T°
ΔH = heat of reaction per unit mass
h_L = latent heat of vaporization per unit mass
I = setting of initial condition pots; see Fig. 2
k = coefficient of thermal conductivity
M_i = symbol for chemical species i
M_s = Mach number of normal shock
m = molecular weight
q_w = heat flux to wall per unit area per second
R = scale factor; see Fig. 2
R_e = Reynolds number
T = temperature
u = mass average velocity in the x direction (wall fixed coordinates)
$\bar{u}$ = mass average velocity in the x direction (shock fixed coordinates)
v = mass average velocity in the y direction
x = distance from shock along the wall
y = distance normal to the wall
Y_i = mass fraction of species i
α = scale factor; see Fig. 2
β = scale factor; see Fig. 2
β_i = defined by Eq. (6)
β_T = defined by Eq. (5)
γ = scale factor; see Fig. 2, also ratio of specific heats
η = similarity parameter defined by Eq. (19)
μ = viscosity
ν_i' = stoichiometric coefficient for species i appearing as a reactant
ν_i'' = stoichiometric coefficient for species i appearing as a product
ρ = density
τ_w = shear stress at the wall
ϕ = fuel to oxidizer mass ratio
ψ = stream function defined by Eq. (18)

Subscripts

1 = conditions upstream of shock
2 = conditions in the convective flow outside the boundary layer
f = fuel species
o = oxidizer species
s = shock
w = wall

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Introduction

IT has recently been discovered that when a shock wave passes over a layer of liquid fuel on the wall of a tube in an oxidizing atmosphere that combustion takes place rapidly enough to reinforce the leading shock wave.[1–4] That is to say, the combustion drives the shock wave in the manner of a detonation. It appears that, initially at least, the combustion takes place within the boundary layer behind the shock wave. While this boundary layer undoubtedly has a low transition-to-turbulence Reynolds number, it is useful to develop a laminar analysis as a basis of comparison with the necessarily less vigorous turbulent analyses. The solution for the laminar case with appropriate assumptions can be obtained from ordinary flat plate boundary-layer theory.

The mathematical formulation of the laminar boundary layer on a stationary flat plate in a uniform air stream was given by Prandtl in 1904 and the resulting differential equation was solved by Blasius.[5] Emmons and Leigh[6,7] extended the formulation and solution of the Blasius boundary-layer equation for flow over a flat plate to include mass addition from the plate and combustion within the boundary layer. Mirels[8] obtained the solution of the Blasius equation for the laminar boundary layer behind a shock wave in a shock tube.

The present solution combines the aforementioned works in order to study the laminar boundary layer on a plate with a thin layer of fuel which is exposed to an oxidizing atmosphere and swept over by a normal shock of constant strength. The model for the problem is shown in Fig. 1. The analysis which leads to the Blasius equation with more general boundary conditions than have previously been used is presented along with the solutions. Coefficients of mass transfer, drag, and heat transfer to the wall may be obtained from the initial conditions. The complete solutions are of interest for future analysis of the boundary-layer profiles.

The mathematical problem is formulated in a coordinate system fixed with respect to the shock. However, the transfer coefficients are defined in terms of the convective velocity relative to the wall. The model for the problem is shown in Fig. 1 in a coordinate system fixed with respect to the shock. The transformation from the laboratory coordinate x to the shock fixed coordinate $\bar{x}$ is given by $\bar{x} = u_s t - x$. The velocity transformation between coordinates is $\bar{u} = u_s - u$.

The following assumptions are made. 1) The flow is laminar, steady, at constant pressure and the usual boundary-layer approximations hold. 2) The Prandtl number is unity, the Schmidt number based on binary diffusion coefficients for each pair of species is unity; body forces, radiative energy transport, and thermal diffusion are neglected. 3) $\rho\mu/\rho_2\mu_2 = 1$. 4) There is a one-step chemical reaction of the form

$$\sum_{i=1}^{N} \nu_i' M_i \rightarrow \sum_{i=1}^{N} \nu_i'' M_i$$

5) The temperature of the vaporizing fuel is constant and equal to the equilibrium boiling point temperature. 6) The properties of the external stream are constant.

Formulation of the Boundary-Layer Equations

Under the preceding assumptions the boundary-layer equations in the form given by Williams,[9] which is referred to as the Shvab-Zeldovich form, are

$$(\partial \rho \bar{u}/\partial \bar{x}) + (\partial \rho v/\partial y) = 0 \tag{1}$$

$$\rho \bar{u}(\partial \bar{u}/\partial \bar{x}) + \rho v(\partial \bar{u}/\partial y) = (\partial/\partial y)[A(\partial \bar{u}/\partial y)] \tag{2}$$

$$\rho \bar{u}(\partial \beta_T \partial \bar{x}) + \rho v(\partial \beta_T/\partial y) = (\partial/\partial y)[A(\partial \beta_T/\partial y)] \tag{3}$$

$$\rho \bar{u}(\partial \beta_i/\partial \bar{x}) + \rho v(\partial \beta_i/\partial y) = (\partial/\partial y)[A(\partial \beta_i/\partial y)] \tag{4}$$

where

$$\beta_T \equiv \frac{\int_{T_0}^{T} C_p dT + \frac{\bar{u}^2}{2}}{\sum_{i=1}^{N} h_i^\circ m_i(\nu_i' - \nu_i'')} - \frac{Y_1}{m_1(\nu_1'' - \nu_1')} \tag{5}$$

$$\beta_i \equiv \frac{Y_i}{m_i(\nu_i'' - \nu_i')} - \frac{Y_1}{m_1(\nu_1'' - \nu_i')} \tag{6}$$

$$A \equiv \rho D_i = \mu = k/C_p \tag{7}$$

and index 1 may represent any particular species; however, it is convenient to have it represent the oxidizer.

In this formulation the reaction rate terms have been eliminated at the expense of obtaining a complete solution for the temperature and composition profiles. However, the velocity profile and the evaporation rate of the fuel layer may be obtained without further loss of generality. This formulation will be sufficient for our present purpose.

The boundary conditions for the energy equation are given by an energy balance at the liquid layer. At $y = 0$ enough heat is conducted from the gas to the liquid to vaporize the fuel leaving the surface and to supply the heat lost from the surface to the interior. Assuming a steady-state composition profile, and that a single chemical constituent is injected at the surface into the gas mixture, and assuming that the liquid surface temperature is maintained at the equilibrium boiling point temperature of the fuel and no surface reactions occur, it follows that[6]

$$(k/c_p)(\partial h/\partial y)_w = (\rho v)_w(h_L + Q)$$

where h_L is the latent heat of vaporization of the fuel and Q is the heat lost from the surface to the interior per unit mass of fuel gasified. Strictly speaking, this is correct only when the value of h_{i_w} is nearly the same for each specie. For convenience Q will be set equal to zero although it may easily be added to the heat of vaporization for a particular problem. At $y = \infty$, $h = h_2$, which is given by the normal shock relations.

In order to write these boundary conditions in terms of β_T it will simplify matters to assume $\partial Y_1/\partial y = 0$ at $y = 0$. Further it will be convenient to take $Y_1 = 0$ at $y = 0$. This implies that all reactions are completed in the gas and none of the oxidizer reaches the wall. This approximation is often used in combustion, but should be examined for a particular problem.

In terms of β, the boundary conditions become

at $y = 0$

$$\beta_T = \beta_{T_w} \tag{8}$$

at $y = 0$

$$\sum_{i=1}^{N} h_i^\circ m_i(\nu_i' - \nu_i'')\left(\frac{\partial \beta_T}{\partial y}\right)_w - u_s\left(\frac{\partial \bar{u}}{\partial y}\right)_w = \frac{C_{p_w}(\rho v)_w h_L}{k_w} \tag{9}$$

at $y = \infty$

$$\beta_T = \beta_{T_2} \tag{10}$$

Boundary conditions on the velocity are at $y = \infty$, $\bar{u} = \bar{u}_2$, which is given by the normal shock relations; and at $y = 0$, $\bar{u} = u_s$, which is the shock velocity. Also at $y = 0$, v must be determined by the vaporization and moving wall conditions as follows. Since the energy equation and the momentum equation have the same form, a particular solution for the energy in terms of the velocity is given by the "Crocco relation,"

$$\beta_T = \left(\frac{\beta_{T_2} - \beta_{T_w}}{\bar{u}_2 - u_s}\right)\bar{u} + \frac{\beta_{T_w}\bar{u}_2 - \beta_{T_2}u_s}{\bar{u}_2 - u_s} \tag{11}$$

and differentiating

$$\partial \beta_T/\partial y = (\beta_{T_2} - \beta_{T_w}/\bar{u}_2 - u_s)\, \partial \bar{u}/\partial y \tag{12}$$

Substituting Eq. (12) into Eq. (9) it is seen that at $y = 0$, v must satisfy the equation

$$\frac{\sum_{i=1}^{N} h_i^\circ m_i(\nu_i' - \nu_i'')(\beta_{T_2} - \beta_{T_w}) - \bar{u}_2 u_s + u_s^2}{\bar{u}_2 - u_s}\left(\frac{\partial \bar{u}}{\partial y}\right)_w = \frac{(\rho v)_w h_L C_{p_w}}{k_w} \tag{13}$$

In order to solve the momentum equation the first step is to transform the equation to incompressible form by applying the Howarth transformation,

$$z = \int_0^y \rho dy \tag{14}$$

$$w = \rho v + \bar{u} \int_0^y \left(\frac{\partial \rho}{\partial \bar{x}}\right) dy \tag{15}$$

Then Eqs. (1) and (2) can be shown to reduce to

$$\partial \bar{u}/\partial \bar{x} + \partial w/\partial z = 0 \tag{16}$$

$$\bar{u}(\partial \bar{u}/\partial \bar{x}) + w(\partial \bar{u}/\partial z) = \rho_2\mu_2(\partial^2 \bar{u}/\partial z^2) \tag{17}$$

Next introduce a stream function ψ and a similarity pa-

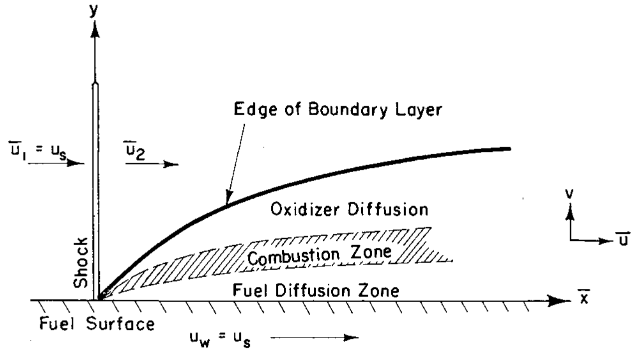

Fig. 1 Boundary-layer model and coordinate system fixed with respect to the shock.

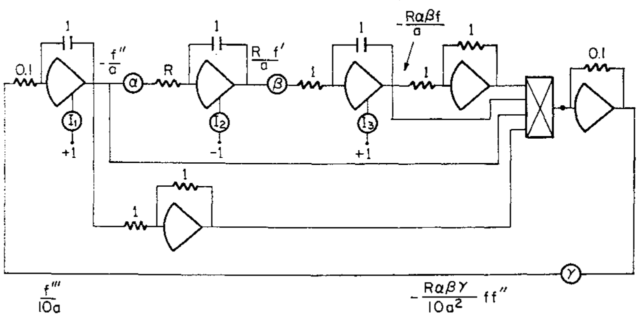

Fig. 2 Circuit used on the analog computer.

rameter η,

$$\psi = (\bar{u}_2\rho_2\mu_2\bar{x})^{1/2}f(\eta) \tag{18}$$

$$\eta = z/2[(\bar{u}_2/\rho_2\mu_2\bar{x})]^{1/2} \tag{19}$$

Then,

$$\bar{u} = \tfrac{1}{2}\bar{u}_2(df/d\eta) \tag{20}$$

$$\rho v = \frac{1}{2}\left(\frac{\bar{u}_2\rho_2\mu_2}{\bar{x}}\right)^{1/2}\left(\eta\frac{df}{d\eta} - f\right) - \frac{1}{2}\bar{u}_2\frac{df}{d\eta}\int_0^y \frac{\partial\rho}{\partial\bar{x}}dy \tag{21}$$

and we obtain the Blasius equation,

$$d^3f/d\eta^3 + f(d^2f/d\eta^2) = 0 \tag{22}$$

with the boundary conditions

$$(df/d\eta)_w = 2u_s/\bar{u}_2 \tag{23}$$

$$(df/d\eta)_\infty = 2 \tag{24}$$

$$[f_w/(d^2f/d\eta^2)_w] = -[B/2(1 - u_s/\bar{u}_2)] \tag{25}$$

where

$$B = \frac{\sum h_i^\circ m_i(\nu_i' - \nu_i'')\,(\beta_{T_2} - \beta_{T_w}) - \bar{u}_2u_s + u_s^2}{h_L} \tag{26}$$

or substituting the definition of β_T from Eq. (5) and denoting the freestream oxidizer by the subscript o the expression for B becomes,

$$h_LB = \int_{T^\circ}^{T_2} C_pdT - \int_{T^\circ}^{T_w} C_pdT + \frac{\bar{u}_2^2}{2} + \frac{u_s^2}{2} - \bar{u}_2u_s + \frac{Y_{o_2}\sum_{i=1}^{N} h_i^\circ m_i(\nu_i' - \nu_i'')}{m_o(\nu_o' - \nu_o'')} \tag{27}$$

The last term of Eq. (27) may be rewritten as

$$Y_{o_2}\left[\frac{m_f(\nu_f' - \nu_f'')}{m_o(\nu_o' - \nu_o'')}\right]\left[\frac{\sum_{i=1}^{N} h_i^\circ m_i(\nu_i' - \nu_i'')}{m_f(\nu_f' - \nu_f'')}\right]$$

The first bracket is the fuel to oxidizer ratio, ϕ, whereas the second bracket is the heat of reaction, ΔH. Finally, assuming constant specific heats the expression for B reduces to

$$h_LB = C_{p_2}T_2 - C_{p_w}T_w + Y_{o_2}\phi\Delta H + \tfrac{1}{2}u_2^2 \tag{28}$$

The coefficients of mass addition, drag, and heat transfer as a function of B and $u_s(u_2)^{1/2}$ may be obtained from the solution of the initial conditions for f and f''. It should be noted that q_w is related to τ_w by the Reynolds analogy which holds rigorously under the assumptions previously listed. This can be shown as follows. Differentiating Eq. (5) and equating it to Eq. (12) yields

$$\frac{\dfrac{\partial h}{\partial y} + \bar{u}\dfrac{\partial\bar{u}}{\partial y}}{\sum_{i=1}^{N} h_i^\circ m_i(\nu_i' - \nu_i'')} = \left(\frac{\beta_{T_2} - \beta_{T_w}}{\bar{u}_2 - u_s}\right)\frac{\partial\bar{u}}{\partial y} \tag{29}$$

Substituting the definition of β_T and rearranging,

$$\frac{\partial h}{\partial y} + \bar{u}\frac{\partial\bar{u}}{\partial y} = \frac{\partial\bar{u}}{\partial y}\left[C_{p_2}T_2 + \frac{\bar{u}_2^2}{2} - \frac{Y_{o_2}\sum_{i=1}^{N} h_i^\circ m_i(\nu_i' - \nu_i'')}{m_i(\nu_i'' - \nu_i')} - C_{p_w}T_w - \frac{u_s^2/2}{\bar{u}_2 - u_s}\right] \tag{30}$$

Table 1 Summary of the constants used with the analog circuit shown in Fig. 2

Case	a	R	α	β	γ	I_1	I_2	$-I_3$
1	1.5	1	0.7500	0.2000	1	−88.53	0	0
2	1.5	1	0.7500	0.2000	1	−9.473	0	10.00
3	3	1	0.7500	0.4000	1	96.08	100.0	0
4	3	1	0.7500	0.4000	1	59.19	100.0	10.00
5	3	1	0.7500	0.4000	1	32.09	100.0	20.00
6	1.5	1	0.3750	0.4000	1	30.09	100.0	30.00
7	1.5	1	0.3750	0.4000	1	19.10	100.0	35.00
8	1.5	1	0.3750	0.4000	1	11.35	100.0	40.00
9	1.5	1	0.2750	0.4000	1	4.90	100.0	47.29
10	8	0.1	0.1333	1	0.6000	84.90	100.0	0
11	6	1	1	1	0.6000	27.80	100.0	50.00
12	6	1	1	1	0.6000	19.35	100.0	60.00
13	6	1	1	1	0.6000	12.95	100.0	70.00
14	6	1	1	1	0.6000	8.24	100.0	80.00
15	6	1	1	1	0.6000	5.08	100.0	90.00
16	12	0.1	0.1500	1	0.8000	95.75	100.0	0
17	6	1	0.7500	1	0.8000	34.75	100.0	50.00
18	6	1	0.7500	1	0.8000	21.50	100.0	60.00
19	6	1	0.7500	1	0.8000	16.64	100.0	65.00
20	6	1	0.7500	1	0.8000	12.70	100.0	70.00
21	6	1	0.7500	1	0.8000	7.08	100.0	80.00
22	20	0.1	0.2000	1	1	84.45	100.0	0
23	10	1	1	1	1	36.29	100.0	40.00
24	10	1	1	1	1	21.23	100.0	50.00
25	10	1	1	1	1	11.54	100.0	60.00
26	10	1	1	1	1	5.80	100.0	70.00
27	10	1	1	1	1	4.00	100.0	75.00

Table 2 Summary of initial conditions satisfying the Blasius equation for various values of M_s and B

Case	$-f(0)$	$f'(0)$	$-f''(0)$	B	M_s
1	0	0	−1.33	0	...
2	1.00	0	−0.14	14.2	...
3	0	4	2.88	0	1.58
4	1.00	4	1.77	1.1	1.58
5	2.00	4	0.96	4.1	1.58
6	3.00	4	0.45	13.3	1.58
7	3.50	4	0.29	24.2	1.58
8	4.00	4	0.17	47.1	1.58
9	4.73	4	0.073	130	1.58
10	0	6	6.79	0	2.24
11	3.00	6	1.67	7.2	2.24
12	3.60	6	1.16	12.4	2.24
13	4.20	6	0.78	21.5	2.24
14	4.80	6	0.49	39.2	2.24
15	5.40	6	0.30	72.0	2.24
16	0	8	11.49	0	3.16
17	4.00	8	2.08	11.5	3.16
18	4.80	8	1.29	22.3	3.16
19	5.20	8	1.00	31.2	3.16
20	5.60	8	0.76	44.2	3.16
21	6.40	8	0.42	91.4	3.16
22	0	10	16.89	0	5.00
23	4.00	10	3.63	8.8	5.00
24	5.00	10	2.12	18.9	5.00
25	6.00	10	1.15	41.7	5.00
26	7.00	10	0.58	96.6	5.00
27	7.50	10	0.40	150	5.00

Since $q_w = -\mu_w (\partial h/\partial y)_w$, Eq. (30) becomes

$$q_w = -\tau_w\left[\frac{C_{p_2}T_2 - C_{p_w}T_w + \frac{1}{2}(\bar{u}_2 - u_s)^2 + Y_{o_2}\phi\Delta H}{\bar{u}_2 - u_s}\right] \tag{31}$$

Integrating q_w and τ_w over $\bar{x}$ and dividing by $\bar{x}\rho_2 u_2$,

$$\frac{\int_0^{\bar{x}} q_w d\bar{x}}{\bar{x}\rho_2 u_2} = \frac{C_D}{2}\left[C_{p_2}T_2 + \frac{u_2^2}{2} + Y_{o_2}\phi\Delta H - C_{p_w}T_w\right] \tag{32}$$

or,

$$C_H = C_D/2 \tag{33}$$

where the mean transfer coefficients are defined as

$$C_M = \int_0^{\bar{x}} \rho_w v_w d\bar{x}/\bar{x}\rho_2 u_2 \tag{34}$$

$$C_D = \mu_w \int_0^{\bar{x}} \left(\frac{\partial u}{\partial y}\right)_w d\bar{x}\Big/\bar{x}\tfrac{1}{2}\rho_2 u_2^2 \tag{35}$$

$$C_H = \frac{\int_0^{\bar{x}} q_w d\bar{x}}{\bar{x}\rho_2 u_2\left(C_{p_2}T_2 + \frac{u_2^2}{2} + Y_{o_2}\phi\Delta H - C_{p_w}T_w\right)} \tag{36}$$

Next it is convenient to introduce a Reynolds number defined following Glass and Hall[10] such that

$$Re_2 = (\rho_2 u_2^2 x)/(\mu_2 \bar{u}_2) \tag{37}$$

The transfer coefficients may be expressed in terms of the Blasius function by introducing Eqs. (20, 21, and 39) into (33–35)

$$C_M(Re_2)^{1/2} = -f(0) \tag{38}$$

$$C_D(Re_2)^{1/2} = \frac{|f''(0)|}{(u_s/\bar{u}_2 - 1)} \tag{39}$$

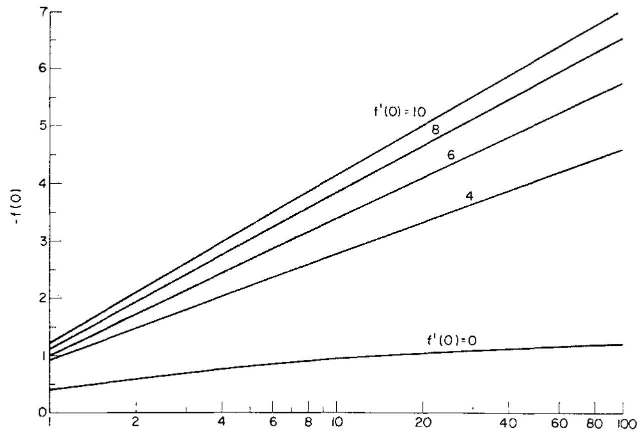

Fig. 3 Initial conditions of the Blasius equation: $f(0)$ vs B for various $f'(0)$.

$$C_H(Re_2)^{1/2} = \frac{|f''(0)|}{2(u_s/\bar{u}_2 - 1)} \tag{40}$$

Solution

The differential Eq. (22) with boundary conditions (Eqs. 23–25) is to be solved. When $u_s = B = 0$ we have the original Blasius equations for a stationary flat plate with no mass additions. When $u_s = 0$ and B is finite we have the Emmons' problem. And when $B = 0$ and u_s is finite we have the shock-tube equations solved by Mirels. Since the differential equation is third order nonlinear, the combined two-parameter set of boundary conditions necessitates that the problem be solved anew. Because we have a two-point boundary value problem in which the starting conditions are not known explicitly for physically interesting cases until the solution is obtained, the equations were solved numerically on an analog rather than a digital computer. The University of Michigan 90 amplifier hybrid analog computer was used with the network shown in Fig. 2. The computer was programed to solve the equations repetitively so that the effect of varying the initial condition pots in order to satisfy the solution at

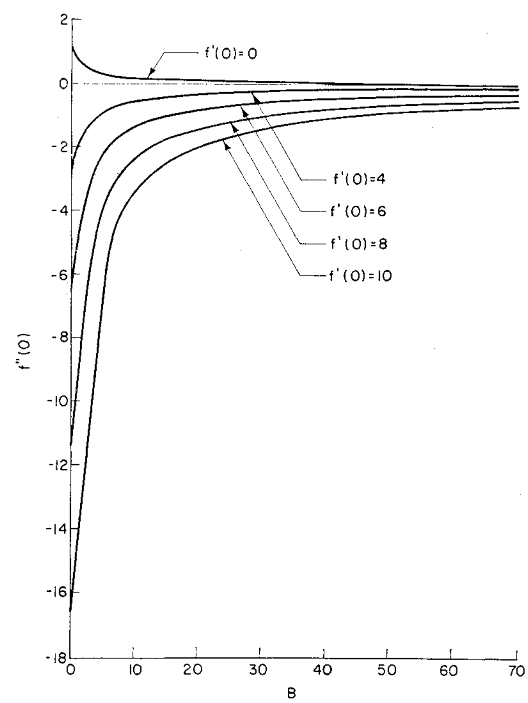

Fig. 4 Initial conditions of the Blasius equation: $f''(0)$ vs B for various $f'(0)$.

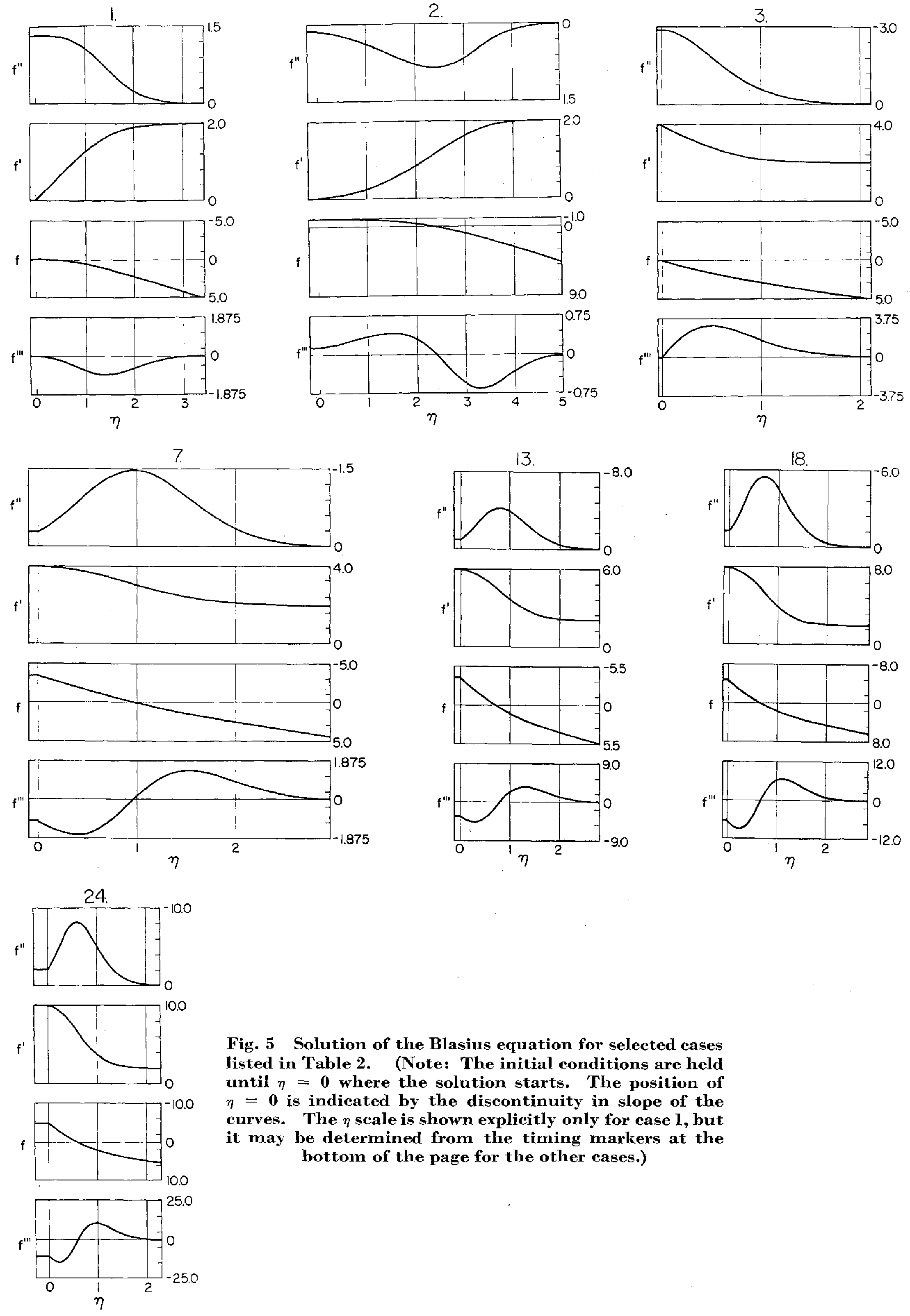

Fig. 5 Solution of the Blasius equation for selected cases listed in Table 2. (Note: The initial conditions are held until $\eta = 0$ where the solution starts. The position of $\eta = 0$ is indicated by the discontinuity in slope of the curves. The η scale is shown explicitly only for case 1, but it may be determined from the timing markers at the bottom of the page for the other cases.)

"infinity" was immediately apparent, and in this manner boundary condition Eq. (25) could be obtained. The initial voltages which were used on the integrating amplifiers and the appropriate scale factors to achieve full-scale deflections are shown in Table 1.

The results of the initial conditions which were obtained as solutions to Eq. (22) are shown in Table 2. The transfer coefficients may be obtained from the initial conditions from Eqs. (38–40). Twenty-seven cases were solved. Cases 1, 2, 3, 10, 16, and 22 were a repeat of previously known solutions in order to provide a check on the technique. Four different velocity ratios ($u_s/\bar{u}_2$), which represented shock Mach numbers of 1.58, 2.24, 3.16, and 5.00 for a gas with a ratio of specific heats of 1.4, were considered. At each velocity ratio the parameter B was varied to cover a range of 0 to at least 70. For convenience $f(0)$ and $f''(0)$ are plotted versus B for the various $f'(0)$'s in Fig. 3 and 4.

The output from the analog computer for selected sets of initial conditions (cases 1, 2, 3, 7, 13, 18, 24 of Table 2) is compiled in Fig. 5. The time axes on the recorder becomes η, and the function f and its three derivatives are recorded as a function of η. The position of $\eta = 0$ is indicated by the discontinuity in slope of the curve. The third derivative is shown to indicate the accuracy of the 37 segment multiplier unit.

Discussion of the Results

As indicated in Figs. 3 and 4, $f(0)$ and $f''(0)$ are very strong functions of the thermodynamic parameter B and also, strong functions of the velocity ratio $f'(0)$. Increasing the enthalpy difference between the freestream and the wall, and increasing the heat release within the boundary layer cause $f''(0)$ to decrease in magnitude towards zero and cause $f(0)$ to increase towards a finite limit. The curves for $f'(0) = 0$ represent the solution obtained by Emmons and Leigh. As pointed out by them, increasing B causes the boundary layer to increase in thickness until finally the boundary layer is "blown off," at which point $f(0)$ approaches the limit of 1.238. The effect of the shock fixed coordinate system is to extend this limit.

As is well known, the effect of mass addition due to vaporization and combustion on a flat plate is to greatly reduce the drag coefficient. The boundary layer behind a shock also exhibits this behavior although to a slightly less degree. The drag coefficient for the shock case is considerably higher than for the stationary flat plate, for the same Reynolds number, however. For a $B = 5$, which is representative of vaporization alone, the drag is reduced by a factor of 3, whereas for a $B = 30$, which is representative of vaporization and combustion, the drag is reduced by a factor of 10.

Referring to Fig. 5, the effect of mass addition is to increase the thickness of the boundary, as evidenced by the fact that f' approaches its asymptotic value at a larger value of η as $f(0)$ is increased. The effect of increasing the velocity ratio $u_s/\bar{u}_2$, or equivalently M_s, is to reduce the boundary-layer thickness. Also, the effect of increasing $f(0)$ is to cause an inflection of the boundary-layer velocity profile. The point of inflection may be located by observing the maximum point of the f'' curve. The effect of increasing the velocity ratio $u_s/\bar{u}_2$ is to suppress this inflection point.

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